Illumination invariants

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Abstract

In this paper we show how tools from group theory can be used to find invariants for time-changing illumination changes. In the first part of the paper we give a brief overview over the theory of one-parameter groups and their invariants. In the second part we describe how the chromaticity coordinates of time-changing illumination spectra can be described by one-parameter curves. Examples for black-body radiators and measured illumination changes are presented. In the last section we describe the invariants for several selected one-parameter groups and the black-body radiators. The derivation of the differential equations and their solution can be done with the help of symbolic mathematical software packages like Maple and we describe a simple Maple session in which the invariants are constructed.

1. Introduction

An invariant is a function that does not change its values under a given collection of transforms. The human visual system employs obviously many mechanisms that are based on invariance principles. Color constancy, ie. the ability to compensate the effects of changing illumination conditions, is a typical example.

In this paper we first introduce invariants in the framework of transformation groups where the set of transformations have a group theoretical structure. We summarize the basic facts from invariant theory, give an overview over the number of invariants and demonstrate how to construct these invariants.

In the color science related part of the paper we show how many color imaging situations can be described in the framework of transformation groups. We first summarize some facts from Principal Component Analysis of spectral distributions that show that spectra can be described by coordinate vectors in a cone. In the case of three-dimensional coordinate vectors the length of the coordinate vector is related to the intensity of the spectra and a two-dimensional vector, located on the unit disk, is related to chromaticity. We then show that relevant sources such as black body radiation and long series of time varying daylight illuminations can be described by one-parameter subgroups. Intensity changes are described by the scaling group and chromaticity changes by subgroups of the group **SU(1,1)**, the symmetry group of the unit disk. In this case we show how invariant theory can be used to get a complete overview over all invariants. We furthermore show how to compute these invariants.

2. One-parameter subgroups and invariants

Let us first introduce the necessary mathematical concepts used in the investigation of time-varying spectral distributions. We will describe the main ideas but omit a number of, important, technical details. The interested reader can find the complete description in many textbooks such as [16, 17, 18, 19].

Let W be a finite dimensional real or complex vector space. For a real vector space we denote by K the space of real numbers $K = \mathbb{R}$ and for a complex space $K = \mathbb{C}$. By G we denote a group of transformations acting on W, ie.: $g : W \to W, x \mapsto g\langle x\rangle$ for all $g \in G$. All groups in this paper are matrix groups where the elements $g \in G$ are $n \times n$ matrices. An $G - invariant$ function is a function $f : W \to K$ satisfying:

$$f\left(g\left\langle x\right\rangle\right) = f\left(x\right); \quad \forall x \in W; \quad \forall g \in G; \tag{1}$$

A familiar example is the vector space of the real two-dimensional plane, the transformations are the rotations and an invariant is the length of a vector. In this case we have:

$$\begin{aligned} K &= \mathbb{R}, \mathbb{G} = \mathrm{SO(2)}, \mathbb{W} = \mathbb{R}^2 \\ x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, g = M = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \\ g\langle x\rangle &= Mx \\ f(x) &= \sqrt{x_1^2 + x_2^2} \end{aligned}$$

In the simplest case the group depends only on one parameter. We define a one-parameter subgroup $\mathbf{M}(t)$ as a subgroup of a group G, defined for real values of t having the properties:

$$\begin{aligned} \mathbf{M}(t_1+t_2) &= \mathbf{M}(t_1)\mathbf{M}(t_2)\ \forall t_1,t_2 \in \mathbb{R}, \\ \mathbf{M}(0) &= \mathbf{E}\text{ (identity matrix).} \end{aligned} \tag{2}$$

The group action is the ordinary matrix multiplication, the inverse of $\mathbf{M}(t)$ is $\mathbf{M}(t)^{-1} = \mathbf{M}(-t)$ and the neutral element is $\mathbf{M}(0)$.

In this paper we focus first on group invariant functions in connection with one-parameter subgroups. This is relevant for color image processing since important chromaticity changes are described by one-parameter groups as we will show later. Furthermore, the restriction to one-parameter groups allows us to sketch the main properties of the method with a least amount of technical difficulties. The generalization to more general groups that depend on a finite number of parameters can be found in the books listed in the references.

Thus we consider the function f as $invariant$ if it satisfies:

$$\begin{gathered} f : W \to K, \\ f(\mathbf{M}(t)\,\langle x\rangle) = f(x);\quad \forall x \in W;\quad \forall t \in \mathbb{R};\ , \\ \mathbf{M}(t)\text{ is an one-parameter subgroups of } G \end{gathered} \tag{3}$$

An invariant function under a one-parameter subgroup can be considered as a function of the variable t. Since it is invariant under variations of t it must be a solution of the differential equation:

$$T_{\mathbf{M}}f = \frac{d}{dt} f(\mathbf{M}(t) < x >)|_{t=0} = 0 \tag{4}$$

A generalization to more general groups can be obtained via infinitesimal generators and Lie algebras: for a one-parameter subgroup $\mathbf{M}(t)$ we define its infinitesimal generator as the matrix $\mathbf{X}$:

$$\mathbf{X} = \frac{d\mathbf{M}(t)}{dt}\ |_{t=0} = \lim_{t\to 0} \frac{\mathbf{M}(t) - \mathbf{E}}{t}. \tag{5}$$

Conversely, we can construct a one-parameter subgroup $\mathbf{M}(t)$ from a given infinitesimal generator $\mathbf{X}$ using the exponential map:

$$\mathbf{M}(t) = e^{t\mathbf{X}} = \mathbf{E} + t\mathbf{X} + \frac{t^2}{2!}\mathbf{X}^2 + ... + \frac{t^k}{k!}\mathbf{X}^k + ... \tag{6}$$

where $\mathbf{E}$ is the identity matrix. The infinitesimal matrices $\mathbf{X}$ form the Lie algebra.

A generalization of one-parameter groups are groups that depend on finitely many parameters. In our case it is sufficient to consider groups with group elements of the form

$$\mathbf{M} = e^{\xi_1\mathbf{X}_1 + \xi_2\mathbf{X}_2 + ... + \xi_n\mathbf{X}_n} \tag{7}$$

The elements $\mathbf{X}_1, \ldots, \mathbf{X}_n$ define an n-dimensional vector space with the additional operation of the Lie product that makes it an n-dimensional Lie-algebra. More information about Lie groups and Lie algebras can be found in the relevant literature, such as [16, 17, 18, 19].

If the group is generated by n matrices $\mathbf{X}_k$ as in Eq.(7) then we get for each one-parameter group

$$\mathbf{M}_k(\xi_k) = \exp(\xi_k\mathbf{X}_k)$$

one differential operator $T_{\mathbf{M}_k} = T_k$ and an invariant function f for the whole group must satisfy the system of partial differential equations:

$$T_k f = 0 \text{ for } k = 1, \ldots n \tag{8}$$

3. Time-changing illuminations and SU(1,1)

In this section we will now show how chromaticity properties of sequences of illumination spectra can be described by one-parameter groups. It is well known that illumination spectra can be described by linear combinations of few basis vectors. [1, 2, 3, 4, 5, 6, 7] Often the eigenvectors of the input correlation matrix are taken as these basis vectors. [2, 8, 9, 10, 11, 12]

We denote in the following a spectral vector by $s(\lambda)$, basis vectors by $b_k(\lambda)$ and the coefficients in the vector σ. We thus have:

$$s(\lambda) \approx \sum_{k=0}^{K} \sigma_k b_k(\lambda). \tag{9}$$

Under the condition that $b_0(\lambda)$ has only positive entries we found that the vectors σ are located in a cone (see [13]). In the case of three basis vectors ($K = 2$) this cone is given by:

$$\mathcal{H} = \{(\sigma_0, \sigma_1, \sigma_2) : \sigma_0^2 - \sigma_1^2 - \sigma_2^2 > 0\}.$$

The basis function $b_0(\lambda)$ is non-negative and the coefficient σ_0 is the scalar product of the spectrum and $b_0(\lambda)$. The coefficient σ_0 is therefore related to the intensity of the spectrum. The projected coefficients $x = \sigma_1/\sigma_0$ and $y = \sigma_2/\sigma_0$ define a point $z = x + iy$ on the unit disk. Since we factor out the intensity related coefficient σ_0 we can think of z as chromaticity coordinate of $s(\lambda)$.

For a sequence of time-changing illumination spectra we get thus a sequence of points on the unit disk that describe the chromaticity changes of the illumination spectra. We will show that such sequences can be described by one-parameter groups and we describe a method how to estimate the one-parameter group from the measurements.

The group **SU(1,1)** consists of all mappings that preserve the hyperbolic geometry (defined by the hyperbolic length and angle) of the disk (more information about hyperbolic geometry can be found in [14, 15]). The hyperbolic distance on the unit disk is given by

$$d_h(z,w) = 2 * \text{arctanh}\frac{|z-w|}{|\bar{z}*w-1|}. \tag{10}$$

and the group **SU(1,1)** if geometry preserving transformations consist of the matrices:

$$\mathbf{SU(1,1)} = \left\{ \mathbf{M} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}; |a|^2 - |b|^2 = 1; a,b \in \mathbb{C} \right\}$$

An element $\mathbf{M} \in$ **SU(1,1)** acts as the fractional transformation on points z on the unit disk:

$$w = \mathbf{M}\langle z \rangle = \frac{az+b}{\bar{b}z+\bar{a}}. \tag{11}$$

The Lie algebra of the Lie group **SU(1,1)** is denoted by *su(1,1)*. It can be shown this Lie algebra forms a three-dimensional vector space [18] with elements of the form

$$\mathbf{X} = \sum_{k=1}^{3} \xi_k \mathbf{J}_k \tag{12}$$

Where the $\mathbf{J}_k$ are given by:

$$\mathbf{J}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \mathbf{J}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

The three real numbers ξ_1, ξ_2 and ξ_3 define the coordinate vector of $\mathbf{X}$ and together with a real parameter t they define a curve on the unit disk as follows: given a starting point $z(0)$ on the unit disk, the coordinates ξ_k and the parameter t we generate first the matrix $\mathbf{X}$ as in (12) and then the curve by:

$$z(t) = \mathbf{M}(t)\langle z(0)\rangle = e^{t\mathbf{X}}\langle z(0)\rangle; t \in \mathbb{R} \tag{13}$$

Given a set of spectra whose chromaticity properties are described by the points $\{z_n = (x_n, y_n); n = 1, ..., N\}$ on the unit disk we developed in [20] two methods to find a one-parameter subgroup connecting these points. We used these methods to fit one-parameter curves to spectra of black-body radiators and measured daylight spectra. The daylight spectra describe spectral distributions in Granada, Spain and Sweden. Figures 1, 2, 3 show the measured chromaticity coordinates and the approximated **SU(1,1)** curves. They show that the chromaticity properties of sequences of illumination changes can be well approximated by **SU(1,1)** curves.

4. SU(1,1) illumination invariants

We now construct invariants for illumination changes. As before we describe the illumination spectrum by a series expansion with coefficients $\sigma_0, \sigma_1, \sigma_2$. Changing the intensity of the illumination source amounts to a simultaneous (positive) scaling of the expansion coefficients and the group action is thus given by

$$(\sigma_0, \sigma_1, \sigma_2) \mapsto e^s (\sigma_0, \sigma_1, \sigma_2) \tag{14}$$

which gives the partial differential equation for the invariant f (D_k denotes partial derivation with respect to variable number k):

$$\frac{df(e^s\sigma_0, e^s\sigma_1, e^s\sigma_2)}{ds} = \sigma_0 D_0 f + \sigma_1 D_1 f + \sigma_2 D_2 f \tag{15}$$

Solving this equation leads to the general solution

$$f(\sigma_0, \sigma_1, \sigma_2) = F(\frac{\sigma_1}{\sigma_0}, \frac{\sigma_2}{\sigma_0}) \tag{16}$$

which means that the general invariant is a function of the ratios σ_k/σ_0.

Next we construct invariants for chromaticity changes. We first consider the simplest cases where the subgroups are given by the three coordinate axis, ie. the differential operators T_1, T_2 and T_3 defined by the coordinate vectors

$$\begin{array}{rcl} & = & (1,0,0) \\ (\xi_1, \xi_2, \xi_3) & = & (0,1,0) \\ & = & (0,0,1) \end{array}$$

This leads to the partial differential equations

$$\begin{array}{ll} PDE_1: & \frac{\partial}{\partial x} f(x,y)(1+y^2-x^2) - 2\frac{\partial}{\partial y} f(x,y)yx \\ PDE_2: & \frac{\partial}{\partial y} f(x,y)(1+x^2-y^2) - 2\frac{\partial}{\partial x} f(x,y)yx \\ PDE_3: & 2\frac{\partial}{\partial y} f(x,y)x - 2\frac{\partial}{\partial x} f(x,y)y \end{array} \tag{17}$$

where (x,y) denote chromaticity coordinates in the unit disk.

For a general one-parameter group with coordinate vector (ξ_1, ξ_2, ξ_3) we obtain the differential equation:

$$\begin{array}{l} PDE: \quad (y^2\xi_1 - x^2\xi_1 - 2y\xi_3 + \xi_1 - 2xy\xi_2)\frac{\partial}{\partial x} f(x,y) \\ \quad +(x^2\xi_2 + 2x\xi_3 - y^2\xi_2 - 2xy\xi_1 + \xi_2)\frac{\partial}{\partial y} f(x,y) \end{array} \tag{18}$$

Solving the partial differential equations (17) gives the following solutions:

$$\begin{array}{ll} Invariant_1: & \mathrm{F}_1\left(\mathrm{x}^2+\mathrm{y}^2\right) \\ Invariant_2: & \mathrm{F}_2\left(\frac{\mathrm{y}^2-1+\mathrm{x}^2}{\mathrm{y}}\right) \\ Invariant_3: & \mathrm{F}_3\left(\frac{\mathrm{y}^2-1+\mathrm{x}^2}{\mathrm{x}}\right) \end{array} \tag{19}$$

where the functions F_k are arbitrary functions of their arguments. Solution F_1 shows, for example, that the general invariant under this subgroup is a function of the radius $x^2 + y^2 = |z|^2$ which can be seen directly since the transformations generated by this element in the Lie-algebra are the rotations of the unit disk. The invariant of a general one-parameter group is the solution of differential equation (18) and given by:

$$I = \mathrm{F_g}\left(\frac{\xi_1^2(-\mathrm{x}\xi_2\xi_3 - \mathrm{x}^2\xi_2^2 - \xi_2^2\mathrm{y}^2 + \xi_2^2 - \xi_3^2 + \mathrm{y}\xi_3\xi_1)}{\xi_2^2(-\xi_3 - \mathrm{x}\xi_2 + \mathrm{y}\xi_1)}\right) \tag{20}$$

For the Planck locus chromaticity the estimation procedure resulted in the three parameters $\xi_1 = -3.54, \xi_2 = 0.45, \xi_3 = -1.47$. Plugging this into Equation (20) gives the invariant

$$\mathrm{F_p}\left(\frac{2025\mathrm{x}^2 - 6660\mathrm{x} + 19879 - 52392\mathrm{y} + 2025\mathrm{y}^2}{45\mathrm{x} - 148 + 354\mathrm{y}}\right)$$

From a practical point of view it is interesting to note that the differential equations and their invariants can be automatically constructed with symbolic math programs like Maple (see Appendix).

The number of functionally independent invariants is given by the dimension of the space on which the group operates (in our case three, given by the three coefficients σ_k) and the dimension of the Lie-algebra. If the scaling group and the chromaticity one-parameter group commute, then the Lie-algebra has dimension two and there is one non-trivial invariant. In the case where they do not commute there are only the trivial constants as invariants.

5. Conclusion

We have shown that if a sequence of changing illumination spectra form a one-parameter group, then it is possible to construct all functionally independent functions that are invariant under these illumination changes.

Acknowledgements

The financial support of the CENIIT program at Linköping University and the Swedish Research Council is gratefully acknowledged. The measured daylight data in Granada is obtained from Javier Hernández-Andrés, University of Granada. The measured daylight data from Sweden was provided by SMHI AB, Sweden.

Biography

Reiner Lenz received a diploma in mathematics from the Georg-August University, Göttingen, Germany, the Ph.D. degree in computer engineering from Linköping University Sweden and he is Associate Professor at the Dept. of Science and Technology, Linköping University, Norrköping, Sweden. He is interested in group-theoretical methods in signal processing and color and multispectral image processing. Thanh Hai Bui had his M.Sc. degree in applied

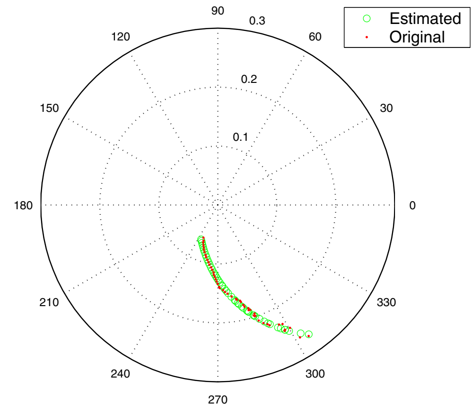

Figure 1: *Granada twilight chromaticity sequence-49 spectra, measured 29/March/1999-and the estimated **SU(1,1)** curve*

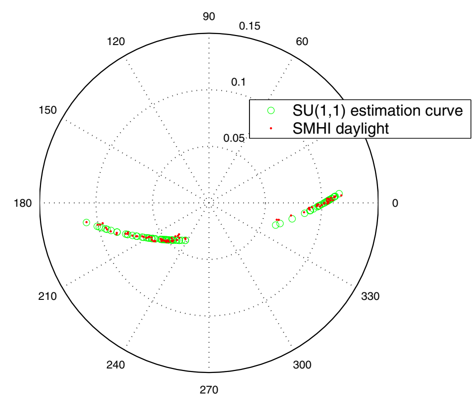

Figure 2: *Swedish daylight chromaticity sequence-79 spectra, measured by SMHI 10/March/1993-and the estimated **SU(1,1)** curve*

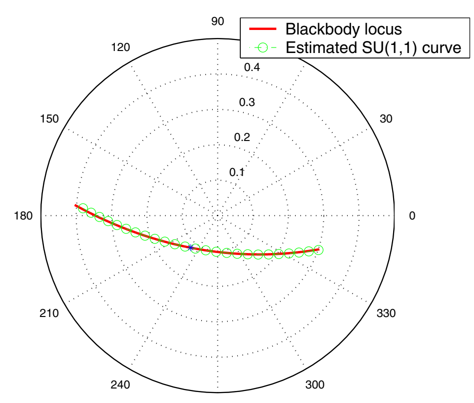

Figure 3: *Blackbody locus-500 spectra ranged [4000..15000K]-and the estimated **SU(1,1)** curve*

computer science from Vrije Universiteit Brussel, he is a Ph.D. student at Linköping University, Sweden.

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A. Maple worksheet for computing invariants

We show here a simple worksheet created in Maple to compute the illumination invariants for **SU(1,1)** one-parameter subgroups mentioned in the text

```
with(linalg):
with(PDEtools):
declare(g(x,y));
declare(f(r,g,b));
im[1] := matrix(2,2,[0,1,1,0]);
im[2] := matrix(2,2,[0,I,-I,0]);
im[3] := matrix(2,2,[I,0,0,-I]);
M := (t,a,b,c) -> evalm(exponential((t*a)*im[1]+(t*b)*im[2]+(t*c)*im[3]));
xfun:=(t,a,b,c,x,y)->evalc(Re(
(M(t,a,b,c)[1,1]*(x+I*y)+M(t,a,b,c)[1,2])/
(M(t,a,b,c)[2,1]*(x+I*y)+M(t,a,b,c)[2,2])));
yfun:=(t,a,b,c,x,y)->evalc(Im(
(M(t,a,b,c)[1,1]*(x+I*y)+M(t,a,b,c)[1,2])/
(M(t,a,b,c)[2,1]*(x+I*y)+M(t,a,b,c)[2,2])));
dxdt0 := subs(t=0,diff(xfun(t,a,b,c,x,y),t)):
dydt0 := subs(t=0,diff(yfun(t,a,b,c,x,y),t)):
xp0 := map(simplify,dxdt0,trig):
yp0 := map(simplify,dydt0,trig):
pdexy := simplify( diff(g(x,y),x)*xp0+diff(g(x,y),y)*yp0,trig);
pde1 := map(simplify,subs(b=0,c=0,pdexy));
pde2 := map(simplify,subs(a=0,c=0,pdexy));
pde3 := map(simplify,subs(a=0,b=0,pdexy));
sol1general := pdsolve({pdexy},[g]);
sol1 := pdsolve({pde1},[g]);
sol2 := pdsolve({pde2},[g]);
sol3 := pdsolve({pde3},[g]);
plancksol := simplify(subs(a=-3.541512,b= 0.456053, c= -1.477389,sol1general));
```