Partial Determination of Non-Linear MTF via the Volterra and Wiener Theories of Non-Linear Systems

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Abstract

The effect of non-linear image processing upon Modulation Transfer Function (MTF) determination in discrete systems is examined and demonstrated. The Volterra and Wiener theories of non-linear systems,[1] which may be considered a generalized form of Linear Systems Theory (LST), are introduced.

The potential of the Wiener theory to aid analysis of discrete imaging devices that contain non-linear processing is explored using a simple computer simulation. It is shown that the majority of the linear component of an MTF may be computed and thus provides a basis for the analysis of systems. Error in the results and expected practical implementational difficulties are discussed.

Problems Associated with Non-Linear Systems

The analysis of imaging systems using LST requires that the system under scrutiny should be linear, spatially invariant and homogeneous.[3] Interesting problems associated MTF measurement occur when the system is non-linear and a number of common examples exist. Transformation of output units into effective exposure units is used to correct for the effects of large area tone reproduction in photographic systems.[4] Development processes have been detailed and modelled using the chemical spread function.[4] Further, mathematical derivations commonly assume low contrast test signals where non-linearities are likely to exist.[4]

The behaviour of digital systems has increased interest in the effect of non-linearities. The nature of digital systems allows non-linear signal processing to be easily included. An example is gradient based sharpening.[5] The strength of sharpening applied is proportional to the gradient of the edge detected. This avoids problems associated with amplifying noise.[5] Presently, artificial intelligence is increasingly being used to enhance the subjective appeal of imaging results. The current trend is an increasing move from passive to active camera agents.

Non-linearities in any imaging system render it scene dependent.[6] MTF measurement therefore depends on the test target and method of analysis used. Figure 1 shows the effect of varying edge target contrast upon a simulated system with gradient based edge sharpening. The system has linear tone reproduction and a Gaussian point spread function (PSF).

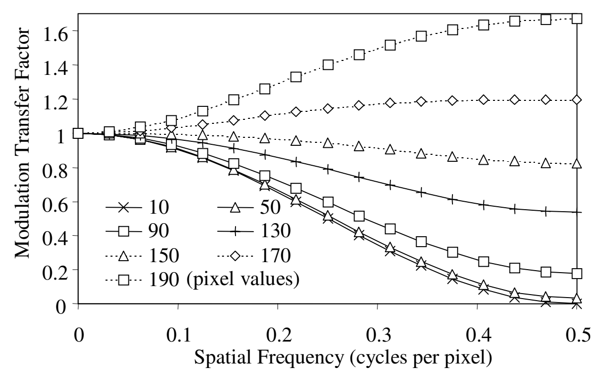

Figure 1. MTF of a system with non-linear sharpening. Each curve represents a test edge of differing magnitude.

A substantial problem is which response may be considered *correct*. All responses of the system are genuine and thus no one may be selected as more or less significant than any other. The result does not aid analysis or further design of system components.

A Simplified Approach to Non-Linear Analysis

The approach to date for imaging system analysis has been to minimize system non-linearities by manipulation of exposure and, for photographic systems, development conditions.[4] It may be argued that this is equivalent to measuring a *linear component* of the system MTF. Design, based on this stable component, is facilitated. The non-linear behaviour of the system may then be assessed as required.[4]

Extraction of the linear component of system MTF may be simplified by assuming that any non-linearity is proportional to the optical contrast of the scene. The assumption is justified if the typical behaviour of photographic systems[4] and filters applied to digital systems are considered.[5] As the contrast of a test signal is reduced, the effect of any non-linearities will diminish.[4] Extending this, it may be imagined that if the contrast of the signal was reduced to zero, non-linearities would not be invoked. Thus, the idea of a *contrast-less edge* is introduced that would enable measurement of the linear component of MTF directly.

The above is nonsense as it is not physically possible to create a contrast-less edge. The concept, however, may be used to develop a technique to extract the linear component. System MTF may be determined a number of times with edges of differing contrast for a non-linear system as in Figure 1. Rearranging the information in the figure, it is possible to examine the effect of the non-linearity with respect to the magnitude of edge contrast, Figure 2. The curves represent the change in single spatial frequencies as edge contrast increases.

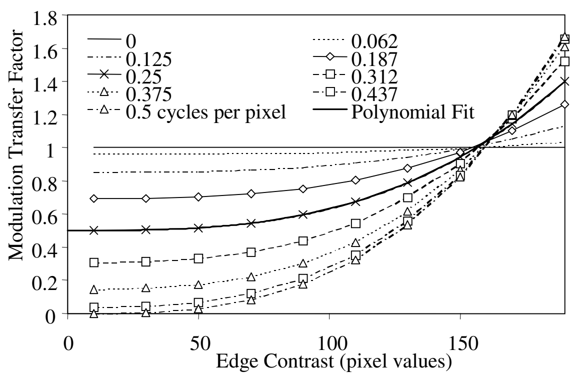

Figure 2. MTF against edge contrast for the data in Figure 1.

For each spatial frequency, the variation in modulation transfer can be seen to be a smooth transition and not random in nature. The exact shape of each curve will depend upon the particular non-linearity present and spatial frequency in question.

The point on the graph at which contrast is zero represents measurement with the *contrast-less edge*. Assuming that the curves continue to change smoothly, it is proposed that interpolation of the results measured with respect to contrast will yield values at this point and thus the linear component of the system MTF. The success of this approach will be affected by the severity of the non-linearity, noise and the type of regression used.

To test the approach, a number of systems were simulated. One dimensional edges of various contrast were created digitally using Microsoft Excel. The edges were convoluted with a Gaussian filter, representing a linear component of the simulated imaging system. For each point of the discrete input, s_n, the output, r_n, was given by:

$$r_n = 0.25s_{n-1} + 0.5s_n + 0.25s_{n+1} \quad (1)$$

The input signal was designed to extend infinitely for the purposes of calculation by replicating the first and last points. Using each edge the MTF was evaluated using a standard approach.[4] It was found that edge contrast had no significant effect upon the determined MTF as all results were equivalent.

A linear sharpening filter, a scaled Laplacian,[5] was additionally applied to each edge. The sharpening filter may be represented by:

$$r_n = -0.6s_{n-1} + 2.20s_n - 0.6s_{n+1} \quad (2)$$

Once the MTF is evaluated, it was again found that the measured response was consistent with respect to edge contrast. A sample curve is shown in Figure 3. A conceptual problem sometimes occurs when MTF curves rise above unity. These points merely represent amplification of the original signal and should not be misinterpreted as the result of a non-linear process. As such these filters do not create a problem for system analysis.

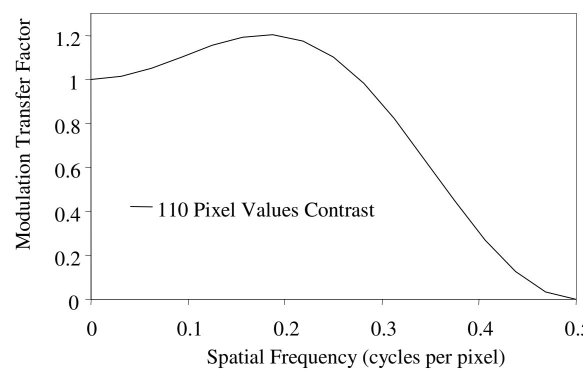

Figure 3. MTF of a system with a linear sharpening filter.

A non-linear sharpening filter may be constructed by scaling the previous filter, with the magnitude of the gradient of the input signal:

$$r_n = s_n + \frac{1}{50}\left[\left|s_{n+1} - s_{n-1}\right| \times \left(-0.6s_{n-1} + 2.20s_n - 0.6s_{n+1}\right)\right] \quad (3)$$

The filter is designed such that there is no sharpening for uniform signals. As edge contrast increases the strength of sharpening applied rises. The constant of 1/50 scales the increase in sharpening. The system MTF may be thought of as comprising of the linear Gaussian component in combination with the non-linear response of the sharpening.

Variation may be seen to exist in the MTF with respect to magnitude of edge contrast used for the test signal, Figure 4. As previously shown, the information may be rearranged with respect to target contrast, Figure 5. The curves may then be interpolated in an attempt to yield the linear component of the MTF, Figure 6.

The Volterra and Wiener Theories

The Volterra and Wiener theories of non-linear systems afford a more sophisticated description of behaviour. An extensive description of the theories is given by Schetzen.[1] An introduction to the theories is summarized from the work of Burns.[2]

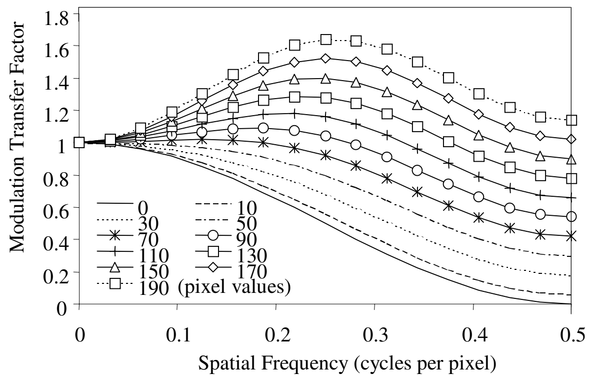

Figure 4. MTF with respect to edge contrast for a system with non-linear sharpening.

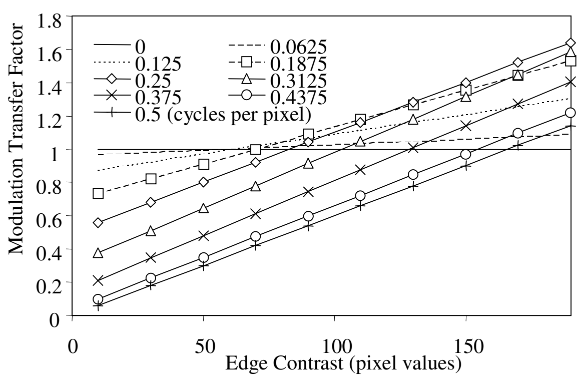

Figure 5. The result of Figure 4 rearranged with respect to test edge contrast.

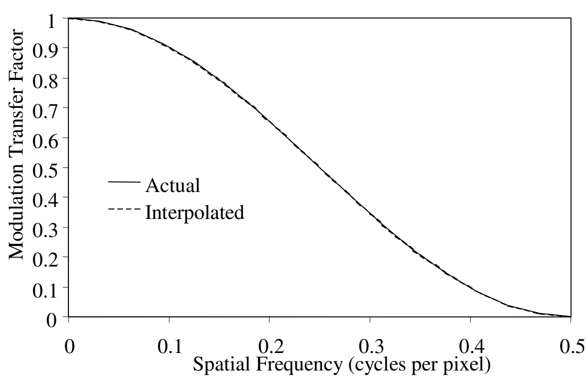

Figure 6. The result of applying regression to the curves in Figure 5 in order to estimate the 'linear' component of the MTF.

Volterra Series

The Volterra description of non-linear system behaviour is a generalized functional series.[2] The terms of the series are n-dimensional convolution integrals based on n-dimensional Volterra kernels.[1,2] For a one-dimensional stationary system,

$$\begin{aligned} r(x) &= h_0 + \int_{-\infty}^{\infty} h_1(\tau) s(x-\tau)\delta\tau \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2) s(x-\tau_1) s(x-\tau_2)\delta\tau_1\delta\tau_2 \\ &+ \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} h_n(\tau_1,\dots,\tau_n) s(x-\tau_1)\dots s(x-\tau_n)\delta\tau_1\dots\delta\tau_n \end{aligned} \tag{4}$$

where *r(x)* is output, *s(x)* input, h_n the set of Volterra kernels and τ_n offset variables.[2]

The Volterra functionals rely being able to describe system non-linearity as a power or polynomial series.[2] Increased system non-linearity is modelled by selecting significant terms of the series until *r(x)* describes behaviour to the desired degree of accuracy.[2] The approach is analogous to that of the Fourier series describing a signal. As a signal increases in complexity greater numbers of terms are required to adequately describe it.[2]

The first term of the series represents bias in the DC component of the modelled system that is not dependent on the input signal. The second term of the series is similar to a standard convolution integral for a linear system.[2] The line spread function(LSF) of the imaging system being represented by h_1. The first order term of the Volterra series therefore represents the linear component of the system. Terms beyond this represent the non-linear behaviour of the system. The particular type of non-linearity will define which terms of the series are used to describe the function.

In the same manner that a linear imaging system may be specified by either its LSF or OTF, a non-linear system may be specified by its set of Volterra kernels or their transforms. The transform for kernel h_n is given by:

$$H_n(\omega_1,\dots,\omega_n) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} h_n(\tau_1,\dots,\tau_n) e^{-2\pi i(\omega_1\tau_1+\dots+\omega_1\tau_1)}\delta\tau_1\dots\delta\tau_n \tag{5}$$

where H_n represents the n-dimensional Fourier transform of h_n and ω_n represents spatial frequency. It should be noted that in the field of electronics the Fourier frequency coefficient employed is commonly -1. This is the case for specification of the Volterra and Wiener series in References.[1,2,7] The field of image science uses –2π and for compatibility this has been employed here. The choice of coefficient reflects a frequency scaling and will not affect results if consistency is maintained.

Burns details two examples which give insight into the behaviour of Volterra analysis.[2] The linear part of the system in Figure 7 has an LSF, *a(x)*. Output of the linear component may then be represented by:

$$r(x) = \int_{-\infty}^{\infty} a(\tau) s(x-\tau)\delta\tau \tag{6}$$

The output after the non-linear component of the system may be represented by:

$$r^n(x) = \left[\int_{-\infty}^{\infty} a(\tau)s(x-\tau)\delta\tau\right]^n \tag{7}$$

and alternatively:

$$g_1(x) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} a_1(\tau_1)\ldots a_n(\tau_n)s(x-\tau_1)\ldots s(x-\tau_n)\delta\tau_1\ldots\delta\tau_n \tag{8}$$

Using the above the Volterra kernels of $g_1(x)$ may be found to be given by:

$$h_m(\tau_1,\tau_2,\ldots,\tau_n) = a(\tau_1)a(\tau_2)\ldots a(\tau_n) \tag{9}$$

where $h_m=0$ for $m\neq n$. Burns then gives the corresponding kernel transforms as:

$$H_m(\omega_1,\omega_2,\ldots,\omega_n) = A(\omega_1)A(\omega_2)\ldots A(\omega_n) \tag{10}$$

where $A(\omega)$ is the Fourier transform of $a(x)$.[2] A *pure* non-linearity of order n will only produce Volterra kernels of the same order n.[2]

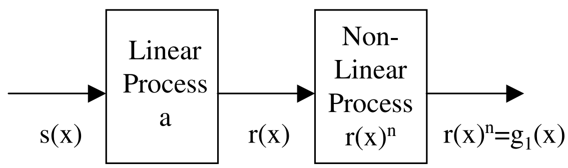

Figure 7. A non-linear system comprising of linear and non-linear components.

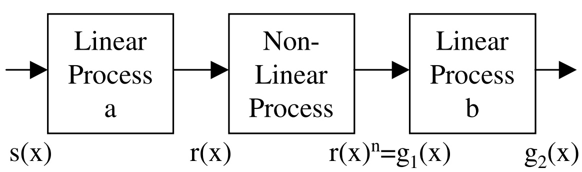

Figure 8. An LNL system comprising a non-linear component 'sandwiched' between linear parts.

The second non-linear system detailed by Burns is commonly referred to as an LNL model,[2] Figure 8. The system consists of a non-linearity *sandwiched* by two linear components.

The LSF of the second linear component is $b(x)$. The output after the second component, $g_2(x)$, may be expressed as:

$$g_2(x) = \int_{-\infty}^{\infty} b(x)g_1(x-\tau)\delta\tau \tag{11}$$

$g_2(x)$ is rewritten as a Volterra series in terms of $g_1(x)$ and is solved to produce the Volterra kernels and their transforms.[2] After a number of steps, it is found that:

$$\begin{aligned} h_n(\tau_1,\ldots,\tau_n) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} b(\tau)a(\tau_1-\tau)\ldots a(\tau_n-\tau)\delta\tau_1\,..\,\delta\tau_n \\ H_n(\omega_1,\omega_2,\ldots,\omega_n) &= B(\omega_1,\ldots,\omega_n)A(\omega_1)\ldots A(\omega_n) \end{aligned} \tag{12}$$

where B(ω) is the Fourier transform of b(x).[2] Significantly, the Volterra kernels are given by combinations of the linear component kernels. The kernel transforms for the LNL system are given by combinations of the linear component transforms. The exact combination is determined by the order of the non-linearity.

Wiener Series

The Volterra series demonstrates how a non-linear system may be represented by combination of n-dimensional Volterra convolution integrals and kernels. Wiener developed the Volterra approach to consider non-linear system response to a white noise signal.[1,2] The Wiener expansion is given by:

$$\begin{aligned} r(x) = k_0 &+ \int_{-\infty}^{\infty} k_1(\tau)s(x-\tau)\delta\tau \\ &+ \int_{-\infty}^{\infty} k_2(\tau_1,\tau_2)s(x-\tau_1)s(x-\tau_2)\delta\tau_1\delta\tau_2 \\ &- C\int_{-\infty}^{\infty} k_2(\tau,\tau)\delta\tau \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_3(\tau_1,\tau_2,\tau_3)s(x-\tau_1)s(x-\tau_2)s(x-\tau_3)\delta\tau_1\delta\tau_2\delta\tau_3 \\ &- 3C\int_{-\infty}^{\infty} k_3(\tau,\tau_1,\tau_1)s(x-\tau)\delta\tau\delta\tau + \ldots \end{aligned} \tag{13}$$

where C is the spectral power of the input signal and k_n the set of Wiener kernels.[2] The expression is summarized by Burns as[2]:

$$r(x) = \sum_{n=0}^{\infty} G_n\left[k_n, s(x)\right] \tag{14}$$

where G_n is the set of Wiener functionals.[2] The Wiener functionals are orthogonal and thus the output of a system may be represented by:

$$r(x) = \lim_{n\to\infty}\left[G_0[k_0,s] + G_1[k_1,s] + \ldots + G_n[k_n,s]\right] \tag{15}$$

The benefit of this representation is that G_n operates only with kernels of order n. Therefore, Burns explains, evaluating additional terms of G_n does not change previously determined terms.[2] Thus, successive Wiener kernels may be determined until a good approximation of system behaviour is reached.[2] Differing non-linearities will produce terms in differing orders of the Wiener series. The first order term of the Wiener series may be thought of representing the linear component of the system. This may be thought of as an alternative method to achieve the separation of the linear component of the system from that which is non-linear.

Determination of Wiener Kernels

Schetzen presents a method for the determination of Wiener kernels using auto-correlation.[1] Burns provides a useful interpretation of the above method.[2] Estimation of Wiener kernel transforms has been shown to require less computation.[1,2] For a white noise input, the results of an extensive derivation show that the first three Wiener kernel transforms of a non-linear system may be estimated as:[1,2]

$$
\begin{aligned}
K_1(\omega) &= \frac{\varepsilon\left[S^*(\omega)R(\omega)\right]}{C} \\
K_2(\omega_1,\omega_2) &= \frac{\varepsilon\left[S^*(\omega_1)S^*(\omega_2)R(\omega_1+\omega_2)\right]}{2C^2} \\
K_3(\omega_1,\omega_2,\omega_3) &= \frac{\varepsilon\left[S^*(\omega_1)S^*(\omega_2)S^*(\omega_3)R(\omega_1+\omega_2+\omega_3)\right]}{6C^3}
\end{aligned}
\tag{16}
$$

for $\omega_1 \neq \omega_2$, $\omega_2 \neq -\omega_3$ and $\omega_1 \neq -\omega_3$. The Fourier transforms of the input and output of the system are denoted *S(x)* and *R(x)* respectively. The use of the complex conjugate is indicated by * and the ensemble average by ε. For spatial frequencies that do not comply with the given conditions, Burns states that an impulse response exists.[2] These points do not represent an accurate estimate of the kernel transforms and are excluded from all results.

Experimental Simulation

Rationale has been presented for extracting the linear component of system response from that which is non-linear. The Wiener series provides this component, manifested as the first order kernel and its transform. Additionally, higher orders provide a description of the non-linear behaviour of the system. Though implementation of the Wiener series is more complex, problems associated with the interpolation of curves plotted with respect to input signal contrast are avoided.

First order Wiener kernel transforms were determined for each of the systems detailed previously. One thousand measurements of each were performed in order to reduce error in the results. Data lengths were maintained at 64 points. Because of the use of the Fast Fourier Transform, 32 useful points are yielded in the results. Calculation of the magnitude of error is presented below and shown as error bars in the resultant curves.

Figure 9 shows a magnitude estimate of the first order kernel transform for the original Gaussian convolution filter. Figure 9 also illustrates the result for the Gaussian and linear sharpening filter in combination.

Figure 10 shows a magnitude estimate of the first order Wiener kernel of the Gaussian and the non-linear sharpening filter in combination. The first order estimate may be seen to be equivalent to that measured for the Gaussian filter applied in isolation. This is as expected as the non-linear sharpening filter was designed to have no linear component of behaviour. The Wiener kernel estimate has successfully separated the linear and non-linear components of behaviour. It may be seen, that for an equivalent number of determinations, the noise is slightly higher in results containing a non-linear component. This effect has also been reported by Burns.[2]

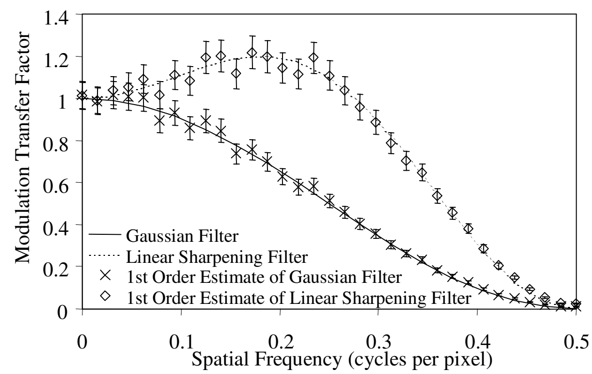

Figure 9. Magnitude estimate of the first order Wiener kernel transforms for the Gaussian filter and linear sharpening filter.

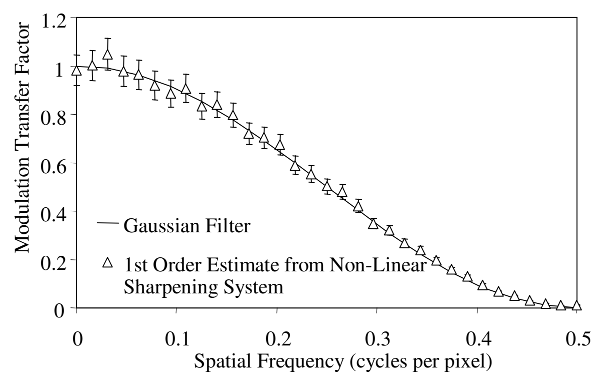

Figure 10. First order magnitude estimate for the system with the non-linear sharpening filter.

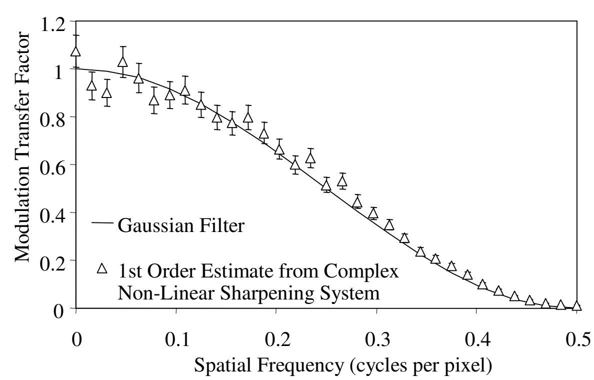

Figure 11. First order magnitude estimate for the system with the complex non-linear sharpening filter.

Figure 11 additionally illustrates that when a complex non-linear sharpening filter is applied in addition to the Gaussian, the Wiener Kernel estimate is still able to successfully extract the linear component of behaviour. The complex non-linear filter was of the form:

$$r_n = s_n + \frac{1}{50}\left[\left|s_{n+1} - s_{n-1}\right| \times \left(-0.6s_{n-1} + 2.20s_n - 0.6s_{n+1}\right)\right] \quad (17)$$

Error in Results

Excluding the points where an impulse response exists, Burns estimates the standard deviation in single kernel transform measurements.[2] He concludes from empirical measurement, that the standard deviation is one, three and nine times the kernel transform magnitude for the first, second and third order Wiener kernels respectively[2].

Taking M measurements, the standard deviation of the averaged result, σ_A, is reduced to:

$$\sigma_A = \frac{\sigma_S}{\sqrt{M}} \quad (18)$$

where σ_s represents the standard deviation in a single measurement. Error bars in the Figures represent $\pm 2\sigma_A$ which includes 95% of values at each point in the final result.

Implementational Difficulties

The ability to extract linear component measures from non-linear systems suggests that long standing issues regarding tone reproduction may be mitigated. i.e. It now becomes unnecessary to transfer output into linear input units before computation. In order to successfully do this, however, all components in an imaging chain need to be analysed using Wieners' method. Also, the ability to design optical components using the Wiener transform results needs to be understood. This is impractical if the system presented in Figure 8 is considered. The non-linear component of this system may be thought of as representing the effect of gamma correction in a typical digital camera. The gamma correction renders the camera entirely non-linear. Wiener kernels of the same order as the power function are generated during analysis. As a consequence, those components traditionally thought of as linear in the MTF of the acquisition device will be arbitrarily displaced into higher orders. Transforming output values of the device being measured into linear input units is, therefore, still of value when performing Wiener analysis. Components that are determined as linear when employing traditional MTF analysis will remain so. In turn it is not necessary to apply Wiener series analysis to all components of an imaging chain as the MTF may be compared with the first order Wiener kernel transform.

The derivation and results presented in this work are for a one-dimensional signal. It may be seen that the n^{th} Wiener kernel involves an n dimensional convolution kernel. Extending the analysis to describe two or three-dimensional systems will require an increase in the order of the Wiener kernels used.

The estimation of the Wiener kernels also requires many measurements using white noise. The value of the test signal at each point needs to be known and aligned precisely with the output of the system. This may deter common application.

Conclusion

Though mathematically complex, the Weiner description of non-linear systems provides an automated method for extracting linear components of frequency response. The technique is particularly suited for analysis of the increasing numbers of non-linear digital imaging systems available and provides significant advantages over previous methods.

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