 

# Analysis of Thread-state Liquid Jet

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## Abstract

It is known that an electrically driven liquid always forms a threadlike stream from the end of which droplets are ejected against the counter electrode in a strong electrostatic field. Careful observation during experiments, however, reveals that the thread-state liquid jet tapers off slightly along the stretched direction. Therefore, it is expected that the electric charge density of the liquid jet driven by a strong electrostatic field can be estimated from a jet figure transformation.

The simplified electro-hydrodynamic equation which describes the liquid jet figure transformation mathematically is derived from Navier-Stokes' equations with some assumptions to give a specific thread-state and an electric motive force.

The derived equation shows a good agreement with the experimentally observed jet form under the limited condition that only the gravitational force works and there is no electrostatic force. Using this equation, a trial is conducted to estimate the charge density of the thread-state water jet.

## Introduction

The droplet-generating process has already been analyzed as a surface wave growth in columnar liquid flow by Lord Rayleigh[1) ], Von C. Weber[2)] and many other scientists. In those analyses, however, the simple assumption is made that the liquid jet has a cylindrical outline and that its radius is constant everywhere. The equation of continuity for incompressible fluids indicates that the constant radius assumption mentioned above also demands that the averaged velocity must be constant at every point in the axial direction.

In an actual liquid jet ejecting process, however, the body of liquid is always accelerated in a strong electrostatic field, and the radius is observed to slightly taper off in the flowing direction. Therefore, the constant radius assumption mentioned above is not allowable for the purpose of detecting the charge density by measuring a very little translation in a liquid jet form.

The theoretical formulation for thread-state liquid flow is needed to clarify the relation between the charge density and the form outline of the liquid jet which is accelerated by gravitational and electrostatic forces continuously.

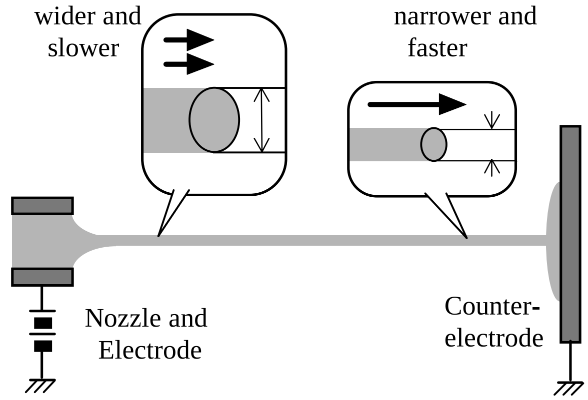


*Figure 1: Simply illustrated thread-state liquid jet.*

## Thread-state formulation

The thread-state formulation will be theoretically deduced here based on the idea that the fluid is enclosed in a slim and long stream tube without breaking into drops. The strong centripetal force to enclose the fluid originates from the surface tension working in a very large curvature which is kept stable by a strong stretching stress in the axial direction.

The following notation is used in the analysis:

| | |
|---|---|
| $r, \theta, z$ | three components of cylindrical coordinate system |
| $\rho$ | density of the liquid |
| $\eta$ | viscosity of the liquid |
| $\gamma$ | surface tension of the liquid |

| | |
|---|---|
| q | constant charge distributed uniformly in the liquid body |
| g | gravitational constant |
| E | electrostatic field intensity |
| Vr, Vθ, Vz | velocity components of liquid |
| p, pr, pz | static pressure and its r and θ components |
| Xr, Xθ, Xz | body force components |

The formulation starts from Navier-Stokes' equations which are equations of motion described for viscid fluids in hydrodynamics. They are expressed for cylindrical coordinates (r,θ,z), as

$$\rho\{(\partial Vr/\partial t)+Vr(\partial Vr/\partial r)+(V\theta/r)(\partial Vr/\partial\theta) -(V\theta)^2/r+Vz(\partial Vr/\partial z)\} = -\partial p/\partial r +\eta\{(\partial^2/\partial r^2)Vr+(1/r)(\partial Vr/\partial r) -(Vr/r^2)+(1/r^2)(\partial^2/\partial\theta^2)Vr -(2/r^2)(\partial V\theta/\partial\theta)+(\partial^2/\partial z^2)Vr\} +Xr \quad \text{(Eq.1)}$$

$$\rho\{(\partial V\theta/\partial t)+Vr(\partial V\theta/\partial r)+(V\theta/r)(\partial V\theta/\partial\theta) +(Vr\, V\theta)/r+Vz(\partial V\theta/\partial z)\} = -\partial p/\partial\theta +\eta\{(\partial^2/\partial r^2)V\theta+(1/r)(\partial V\theta/\partial r) -(V\theta/r^2)+(1/r^2)(\partial^2/\partial\theta^2)V\theta +(2/r^2)(\partial Vr/\partial\theta)+(\partial^2/\partial z^2)V\theta\} +X\theta \quad \text{(Eq.2)}$$

$$\rho\{(\partial Vz/\partial t)+Vr(\partial Vz/\partial r)+(V\theta/r)(\partial Vz/\partial\theta) +Vz(\partial Vz/\partial z)\} = -\partial p/\partial z +\eta\{(\partial^2/\partial r^2)Vz+(1/r)(\partial Vz/\partial r) +(1/r^2)(\partial^2/\partial\theta^2)Vz+(\partial^2/\partial z^2)Vz\} +Xz \quad \text{(Eq.3)}$$

Some simplification is used as follows,

(1) steady state: $\partial Vr/\partial t= 0$ , $\partial V\theta/\partial t= 0$ , $\partial Vz/\partial t= 0$

(2) non rotational: $V\theta= 0$

(3) the axial velocity component (Vz) is uniform in any cross section.: $\partial Vz/\partial r= 0$

(4) the radial velocity component (Vr) is proportional to radius in any cross section. :
$(\partial^2/\partial r^2)Vr =0$ ,
$(1/r)(\partial Vr/\partial r)-(Vr/r^2)=0$

(5) static pressure is constant in any cross section.
$\partial p/\partial r=0$

(6) external field works only in the axial direction.:
$Xr=0,\ X\theta=0,\ Xz\neq 0$

Based on the above assumptions (1) to (6), (Eq.3) is rewritten as

$$\rho\{Vz(\partial Vz/\partial z)\}=-\partial p/\partial z+\eta(\partial^2/\partial z^2)Vz+Xz \quad \text{(Eq.4)}$$

Electrostatic conditions are assumed to be as follows.

(7) The charge density is a constant value in the liquid.

(8) The electrostatic field intensity is also constant everywhere.

(9) The space charge contained in the liquid has no influence on the external applied electrostatic field.

The body force Xz caused by gravitational field and electrostatic field could be described as

$$Xz=qE\pm\rho g$$

Either a plus or a minus sign will be adopted according to whether the fluid flows against or with the gravitational field.

Then (Eq.4) is rewritten as

$$\partial/\partial z\{(1/2)\rho Vz^2\}= -\partial p/\partial z+\eta(\partial^2/\partial z^2)Vz+(qE\pm\rho g) \quad \text{(Eq.5)}$$

After (Eq.5) is indefinitely integrated with respect to z, the rough property of thread-state is added based on the following two assumptions.

(10) The side surface of liquid flow forms an axial-symmetry.

(11) The side surface of liquid flow is the only place where the surface tension works.

Therefore, the static pressure can be related with the surface tension of liquid, as

$$p=(1/3)(pr+pz)=(1/3)\{-(\gamma/r)-(2\pi r\gamma/\pi r^2)\} =-\gamma/r$$

Thus,

$$(1/2)\rho Vz^2-\gamma/r-\eta(\partial Vz/\partial z)-(qE\pm\rho g)z=\text{const.} \quad \text{(Eq.6)}$$

And equation of continuity is

$$\pi r^2 Vz=\text{const.} \quad \text{(Eq.7)}$$

It should noticed from (Eq.6) and (Eq.7) that radius (r), velocity (Vz) and composite potentials compose the balancing system for inviscid fluid (η=0). Running in the field forces direction, the fluid is able to increase its moving energy, but causes only a very small decrement of its radius because the moving energy is inversely proportional to fourth powers of radius in this balancing system. This is why the stable thread-state can be kept without breaking into drops.

## Discussion

To ascertain the reliability of (Eq.6), it is necessary to make a comparison with the actual experimental results. The paper presented by G. I. Taylor[3)] contains many useful data that can be compared with theoretical calculation. In particularly, there are some useful photographs in that paper which show the forms of thread-state liquid jets. If the derived equation (Eq.6) is correct, the characteristics calculated with that equation will be in complete agreement with these experimental results.

First, minimum data is needed to calculate the jet form, such as an inviscid liquid under the conditions that there is no electrostatic field and only gravitational force works. Fortunately, we could pick up, from Taylor's paper, a complete set of data composed of physical parameters and jet form photographs obtained under the conditions mentioned above.

The liquid, which is used in the set of data, is water whose specific resistance is in the order of $10^{8}$ [Ω-cm]. In Figure 2, the theoretically calculated thread-state form is drawn in light line. And the experimentally measured thread-state form shown in the photographs is drawn in heavy line with the error ranges of both upper-end and lower-end data plotted. It should be kept in mind that the transformation of diameter is greatly emphasized because the y scale ratio is 50 times greater than the x scale.

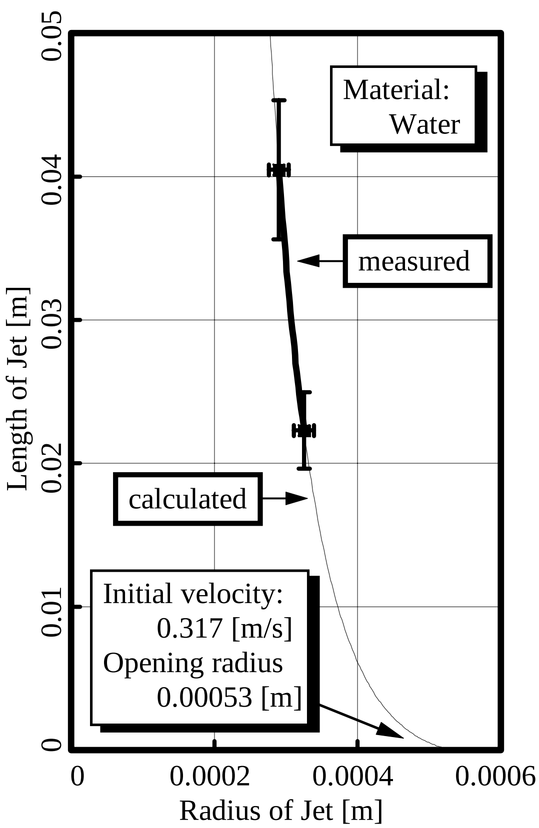


*Figure 2: Thread-state jet form expressed by its length versus radius. Light line is theoretically calculated by (Eq.6). Heavy line is measured from Taylor's photographs by using microscope.*

It can be seen in Figure 2 that the thread-state jet slightly tapers off in the liquid flowing direction, and the experimentally measured form fits well on the theoretically calculated form. Therefore, the usefulness of theoretically derived equation (Eq.6) is considered to be confirmed.

In the next case, the equation is examined under the condition that an electrostatic field is also applied. A charge density expressed by a letter "q" in the potential term in (Eq.6) is treated as an unknown variable factor in actual measured data.

In Figure 3, the five curves which are plotted in light lines are theoretically calculated under the condition of 3.95 x $10^{5}$[V/m] of electrostatic field and the charge densities of zero, 0.1, 0.2, 0.4 and 0.8 [C/$m^3$] are used for "q" parameters. The actual curve which is experimentally measured under the same electrostatic field condition is plotted in heavy line.

The actual charge density can be estimated by tuning the "q" parameter which can be varied adaptively to minimize the fitting error among the calculated curve and the measured curve. In the case of Figure 3, the charge density is estimated to be 0.2 [C/$m^3$].

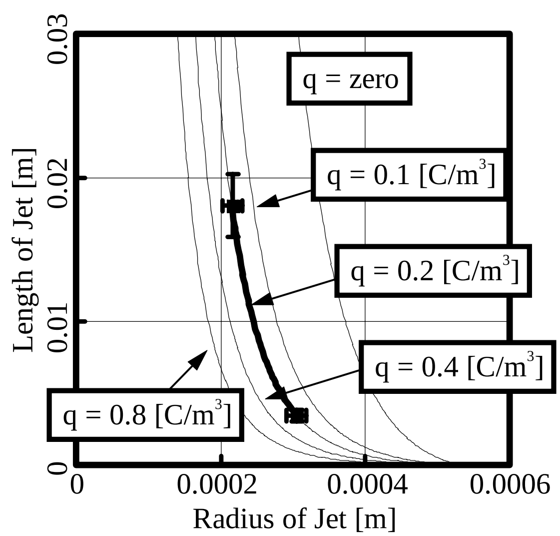


*Figure 3: A group of jet forms calculated for seve-ral "q" parameters. The electrostatic field is con-stant at 3.95 x $10^{5}$ [V/m]. Experimentally measured heavy line shows a good agreement with 0.2[C/$m^3$] charged curve line.*

From this discussion, it is recognized that (Eq.6) also has the ability to estimate the charge density, which is rather difficult to measure directly in the case of an experiment.

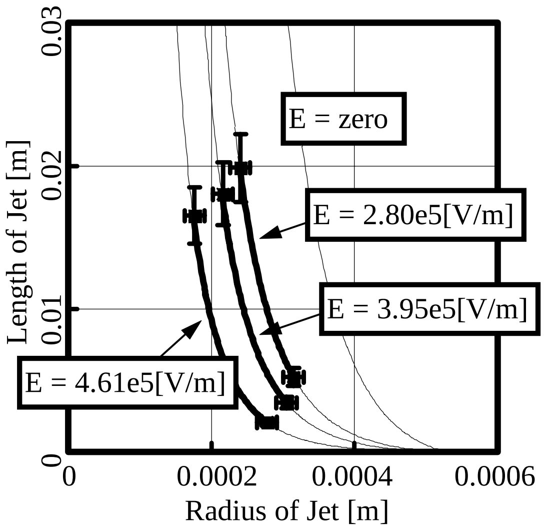


*Figure 4: Electrically driven water jet transforming characteristics. Theoretically calculated light lines and experimentally measured heavy lines are plotted for the parameters of electrostatic field.*

Figure 4 and the next table show the jet forms for various electric field and their charge densities which are estimated by the same procedure as used above. Apparent in Figure 4 is the characteristic whereby the radius of the liquid jet decreases in accordance with the increase of applied electrostatic field.

| Electrostatic field | Estimated charge density |
|---|---|
| $2.80 \times 10^5$ [V/m] | $0.12 \pm 0.04$ [C/m$^3$] |
| $3.95 \times 10^5$ [V/m] | $0.20 \pm 0.05$ [C/m$^3$] |
| $4.61 \times 10^5$ [V/m] | $0.49 \pm 0.12$ [C/m$^3$] |

The results obtained for all samples referred to in this discussion indicate that the theoretical calculation is in good agreement with the experimental result reported by G. I. Taylor. Therefore, the usefulness of theoretical derived (Eq.6) is considered to be confirmed.

Thus, we can easily predict the thread-state form of the liquid jet which is accelerated under the electrostatic field and gravitational field.

## Conclusion

A liquid jet figure formed in an electrostatic field could be expressed by the following equation:

$$(1/2)\rho V z^2-(qE\pm\rho g)z-\gamma/r-\eta(\partial Vz/\partial z)=\mathrm{const}$$

In the limited case that no electrostatic force is applied and only the gravitational force works, this equation has been confirmed to be correct by comparing the calculation results with the experimental data for an inviscid liquid.

Using this equation, the charge density in the thread-state of water is estimated to be about $0.1\sim0.5$ [C/m$^3$] under the electrostatic field condition of $2.8\sim4.6\times10^5$ [V/m].

Further experimental confirmation is required before the derived equation can be considered to be perfectly reliable. However the author expects the simplified equation to be useful in the analysis of and facilitate understanding of fluid motion in an inkjet.

## References

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