# The Effect of Quantization Upon Modulation Transfer Function Determination

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## Abstract

A theoretical treatment of the effect of quantization upon the determination of modulation transfer functions (MTF) of digital acquisition devices is initially developed for an noiseless system, when using sinusoidal targets.

The analytical work shows that a component due to quantization exists in the measured MTF which increases as the bitdepth of the quantization and the amplitude of the input signal decreases. An expression to estimate this component from parameters describing the input signal and quantizer is derived.

Modifications are made to account for quantization and input signal noise, yielding new estimates of the component for single measurements. The estimates were experimentally tested using an analogue to digital converter (ADC).

## Introduction

The purpose of quantization in digital image acquisition systems is to map a continuous range of input intensities to a discrete set of output values which may be subsequently be used in digital calculations. Representation of the input in this manner introduces a quantization error, the difference between the original and quantized signal. The quantization error will cause a change in the measured MTF of the system in question.

A difficulty which exists when performing component analysis upon digital acquisition devices is the separation of the effects of sampling from quantization. Most models consider the change made to a continuous input signal after being sampled and quantized. The purpose of this work is to estimate the variation in measured MTFs due to the quantization process in isolation.

## Theoretical Method

A continuous sinusoidal input, $E(x)$, may be described by, $E(x)=a + b\cos(2\pi\omega x)$, where $a$ is average signal, $b$ the amplitude, $\omega$ the spatial frequency and $x$ distance[1]. The modulation of the signal, $M_{IN}$, is given by:

$$M_{IN} = \frac{E(x)_{MAX} - E(x)_{MIN}}{E(x)_{MAX} + E(x)_{MAX}} \tag{1}$$

where $E(x)_{MAX}$ and $E(x)_{MIN}$ are the maximum and minimum values of the signal [1].

When applied to the input signal, an idealized uniform quantizer may be represented by the following function:

$$Q(x) = Int\left[\frac{E(x)\times(2^d - 1)}{I_{MAX}} + 0.5\right] \tag{2}$$

where $Q(x)$ is the signal after quantization and $d$ the bitdepth of quantization [2]. The quantizer will accept input values between zero and $I_{MAX}$. $Int[\ ]$ represents an integer truncation function. The difference, $\iota$, in terms of linear input units, represented by each quantization level is calculated by:

$$\iota = \frac{I_{MAX}}{2^d - 1} \tag{3}$$

Figure 1 shows the effect of the described quantization function upon $E(x)$. $Q(x)$ has been normalized by multiplication with $\iota$ to lie in the same range as $E(x)$ and no spatial sampling has taken place. The quantization function is non-linear but stationary, thus there are no spatially dependent effects in $Q(x)$. Discontinuities in the output are due to the piecewise continuous nature of the function. This lack of spatial dependency isolates the effect of the quantization process from that of sampling. This work assumes that the MTF of the ADC is negligible over the utilized bandwith.

### Quantized Signal Modulation

The modulation of the quantized signal may be calculated in a similar manner to that of the input:

$$M_Q = \frac{Q(x)_{MAX} - Q(x)_{MIN}}{Q(x)_{MAX} + Q(x)_{MIN}} \tag{4}$$

where $Q(x)_{MAX}$ and $Q(x)_{MIN}$ are the maximum and minimum values of the quantized signal.

For a given bitdepth, the output modulation may be plotted with respect to input signal amplitude or mean input signal level, Figures 2 and 3. The input signal amplitude and mean level are expressed as a percentage of the input range of the ADC, $I_{MAX}$.

The modulation of the quantized signal may be seen to be different from that of the input and displays reasonably complex behaviour. It is shown in both cases the quantized signal modulation oscillates about that of the input. The frequency and magnitude of these oscillations depends upon the bitdepth of the ADC used.

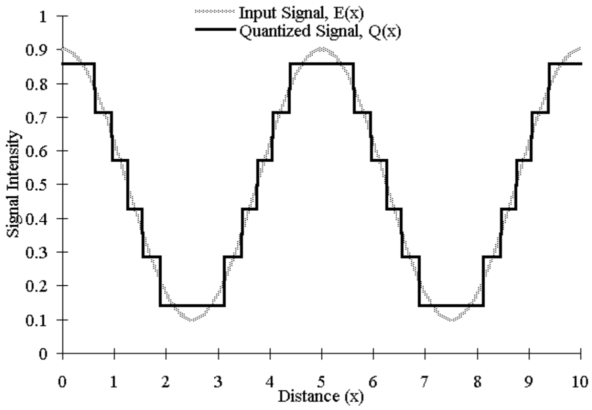


*Figure 1. A sinusoidal input signal of modulation 0.8 (a=0.5, b=0.4) and its corresponding output when quantized using 3 bits. The input range of the quantizer is normalized so that $I_{MAX}=1$.*

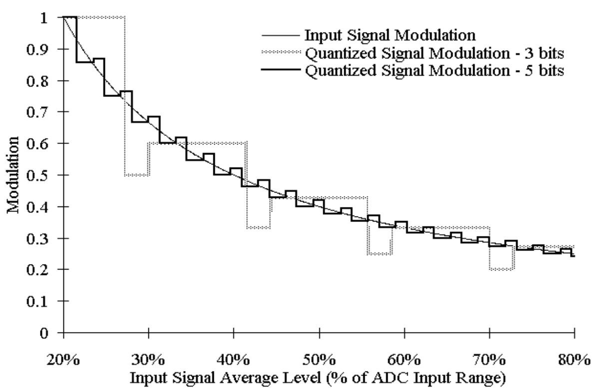


*Figure 2. Input signal modulation and quantized signal modulation versus mean input signal level. The amplitude of the input signal is constant at 20% of the input range of the ADC.*

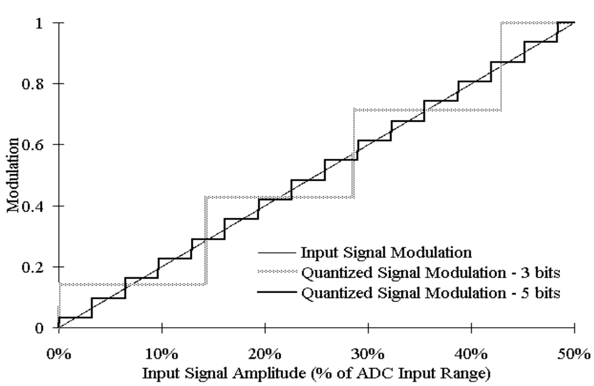


*Figure 3. Input signal modulation and quantized signal modulation versus input signal amplitude. The mean level of the input signal is kept constant at 50% of the input range of the ADC.*

## Quantized Signal Modulation Bounds

The effect of quantization on signal modulation may be judged from graphs such as Figures 2 and 3, given the input signal and quantization parameters. This, however, is impractical in all but a minority of cases, because the actual quantized signal modulation changes rapidly with respect to input signal amplitude and mean level. Thus, a small change in input parameters may yield a large change in the estimate of the effect caused by quantization. It may also be reasonably assumed that the signal modulation reaching the quantization stage of an imaging device is difficult to estimate accurately due to its modification by the optical image forming, spatial sampling and amplification stages.

A better method is to estimate the maximum, rather than actual, effect that quantization may have upon the signal modulation. This is based upon the maximum quantization error and effectively calculates an envelope containing the quantized signal modulation function.

The quantization error, $\varepsilon$, is clearly visible as the difference between the input signal and the quantized output, Figure 1, and may be defined:

$$\varepsilon = E(x) - Q(x) \tag{5}$$

For a perfect quantizer, it may be shown the magnitude of $\varepsilon$ does not exceed $\iota/2$ [3]. Minimum quantization error occurs when the input signal coincides with an ADC decision level and is therefore zero. Given the possible quantization error, the range of values that the quantized signal, $Q(x)$, may take for a given value of the input, $E(x)$, is:

$$E(x) - \frac{\iota}{2} \le Q(x) \le E(x) + \frac{\iota}{2} \tag{6}$$

To calculate the described bounding envelope, $Q_{MAX}$ and $Q_{MIN}$ are substituted in Equation 4 by the extremes of the above range. The upper and lower values of the envelope are then found by rearranging the resulting formulae and solving to yield the maximum and minimum functions. It is found that the upper and lower boundaries of the envelope are then (Figures 4 and 5):

$$M_{Q_{MAX}} = \frac{E(x)_{MAX} - E(x)_{MIN} + \iota}{E(x)_{MAX} + E(x)_{MIN}} \tag{7}$$

$$M_{Q_{MIN}} = \frac{E(x)_{MAX} - E(x)_{MIN} - \iota}{E(x)_{MAX} + E(x)_{MIN}} \tag{8}$$

Given the input signal amplitude, mean level and quantization bitdepth for an ideal quantizer, the quantized signal modulation would be expected to fall between the bounds of the calculated envelope. This approach is improved as it provides limits as to the maximum magnitude of the quantization effect for given circumstances. This simple calculation also enables a better estimate of the significance of the quantization process within the examined imaging system to be made.

In order to better understand the magnitude of the quantization effect on the input signal modulation, the difference between the envelope bounds and the input signal modulation may be expressed as a ratio (or percentage) of the input signal modulation. These modified upper and lower bounds, $\Delta M_{MAX}$ and $\Delta M_{MIN}$, are calculated using:

$$\Delta M_{MAX} = \frac{M_{Q_{MAX}} - M_{IN}}{M_{IN}} \tag{9}$$

$$\Delta M_{MIN} = \frac{M_{Q_{MIN}} - M_{IN}}{M_{IN}} \tag{10}$$

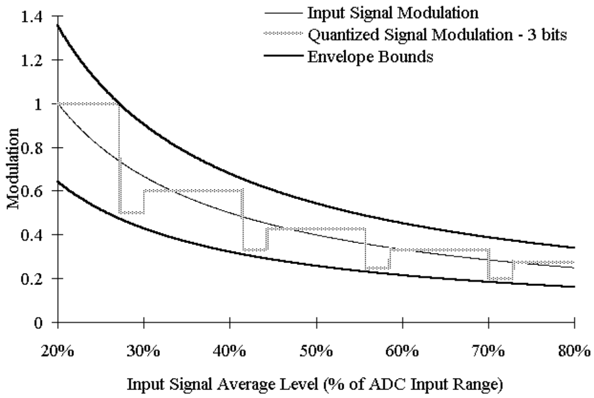


*Figure 4. The calculated envelope versus mean input signal level for a 3 bit ADC. Input signal amplitude is kept constant at 20% of the input range of the ADC.*

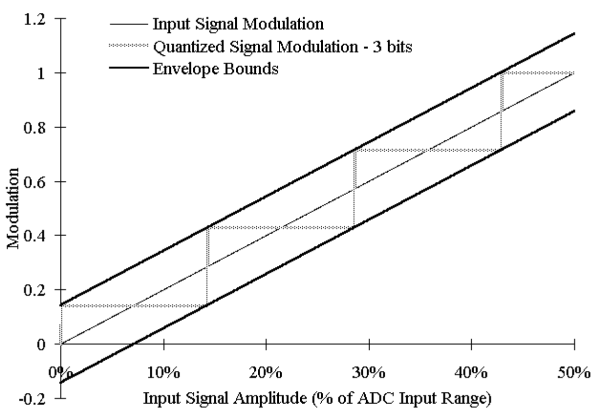


Figure 5. The calculated envelope versus input signal *amplitude for a 3 bit quantization stage. The mean level of the input signal is kept constant at 50% of the input range of the ADC.*

For a specified input signal and quantization bitdepth the percentage deviation from the input modulation caused by quantization would be expected to be between these calculated bounds. It may be shown that the magnitudes of $\Delta M_{MAX}$ and $\Delta M_{MIN}$ are equivalent and may be denoted $\Delta M$. This enables the expected quantized signal modulation to be written as $M_{IN} \pm \Delta M$. Expanding $\Delta M$ it is found:

$$\Delta M = \left| \frac{I_{MAX}}{2(2^d - 1)b} \right| \tag{11}$$

This result is significant as it $\Delta M$ may be seen to be a function of $b$, $d$ and $I_{MAX}$ only. Thus, the percentage change expected in modulation is only dependent upon the input range of the quantizer, the bitdepth of the quantizer and the amplitude of the input signal. Figure 6 shows that the relative change in the input signal modulation increases as the amplitude of the input signal and the bitdepth of the quantization decreases.

The input signal amplitude necessary to reduce the quantization effect to within a specified value may be calculated by rearranging Equation 11 to yield $b$:

$$b = \left| \frac{I_{MAX}}{2(\Delta M_{MAX} - 2^d \Delta M_{MAX})} \right| \tag{12}$$

**Signal and Quantization Noise**

The above is derived for a perfect system, which in practice rarely occurs. The input signal and quantization process will include noise, causing the effect of quantization to be under-estimated for single measurements. The effect of the noise in both cases may be included to yield new bounds for the quantized signal modulation.

It is assumed that noise present in the input signal is Gaussian distributed and ergodic. A convenient way to describe the variation in the input signal that this introduces is to use the $2\sigma$ deviation of the intensity fluctuations, as 95% of values will fall within these limits[4]. Thus, the input signal including noise may be represented as $E(x) \pm 2\sigma$. The $2\sigma$ deviation describing the input signal noise is included in Equation 6 in order to yield a modified range that $Q(x)$ may take for given $E(x)$ with noise of variance $\sigma$, below:

$$E(x) - 2\sigma - \frac{\iota}{2} \le Q(x) \le E(x) + 2\sigma + \frac{\iota}{2} \tag{13}$$

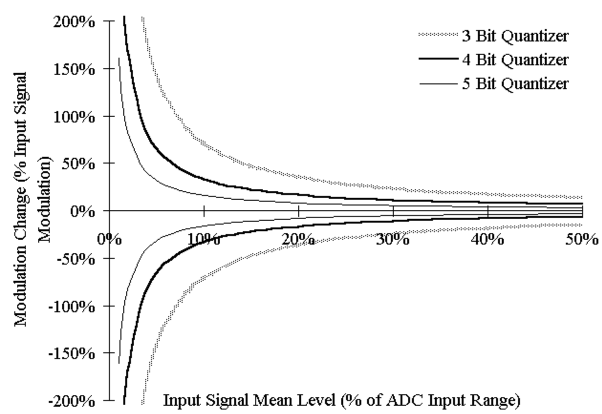


*Figure 6. The calculated envelope versus input signal amplitude for 3, 4 and 5 bit ADCs.*

The internal electronics of the quantization device will generate noise. Thus, it is common practice when purchasing an ADC for the quantization bitdepth to be quoted in combination with an error term for the device. The error term is quoted as a fraction of the least significant bit (LSB). Thus, an eight bit ADC may be described as $d$ bits $\pm\, n$ LSB.

 

The LSB represents a single quantization level, thus the error term may be converted into linear input units simply by multiplication with $\iota$, becoming $\pm n\iota$. Again, in a similar manner to above, this term may be included in Equation 12 to calculate the range of values that Q(x) may take given an input signal with noise of variance $\sigma$ and aquantization stage with noise of $\pm n\iota$:

$$E(x)-2\sigma-n\iota-\frac{\iota}{2}\le Q(x)\le E(x)+2\sigma+n\iota+\frac{\iota}{2} \qquad (14)$$

$Q(x)_{MAX}$ and $Q(x)_{MIN}$ in Equation 4 are substituted for by the extreme limits of the modified range for the real system in a similar manner to before. The resulting equations are solved to find the maximum and minimum bounds of the quantized signal modulation. The calculated bounds for the quantized signal modulation including the effects of noise are then:

$$M_{Q_{MAX}}=\frac{E(x)_{MAX}-E(x)_{MIN}+4\sigma+\iota+2n\iota}{E(x)_{MAX}+E(x)_{MIN}} \qquad (15)$$

$$M_{Q_{MIN}}=\frac{E(x)_{MAX}-E(x)_{MIN}-4\sigma-\iota-2n\iota}{E(x)_{MAX}+E(x)_{MIN}} \qquad (16)$$

These modified bounds, are greater than for the ideal system, thus an increased component in the measured MTF is predicted for a given set of system parameters. It is possible as previously to express these limits as a ratio with respect to the input signal modulation:

$$\Delta M=\left|\left(\frac{I_{MAX}}{2(2^d-1)b}\right)+\left(\frac{nI_{MAX}}{2^d-1}\right)+\left(\frac{2\sigma}{b}\right)\right| \qquad (17)$$

As previously, $\Delta M$ is seen not to be a function of *a*. The input signal amplitude necessary to reduce this component to a required level may again be calculated by rearranging and solving for *b*, below:

$$b=\left|\frac{(-1-2n)I_{MAX}+(1-2^d)4\sigma}{2(\Delta M-2^d\,\Delta M)}\right| \qquad (18)$$

As the values of $n\iota$ and $\sigma$ are fixed for a specific ADC and input signal, the minimum effect due to quantization in the system is increased.

## Experimental and Results

Sinusoidal waves of varying amplitude and a mean level of zero volts were generated using a Farnell LFM4 wave generator. Their amplitude was measured using a Hameg HM203-7 oscilloscope. The variance, $\sigma$, of the input signal noise was estimated using the oscilloscope and recorded as 5.45mV.

The generated signal was fed into a simple analogue to digital conversion circuit, consisting of a TL061 operational amplifier and Ferantti ZN449E ADC[5]. Digitization was controlled and data captured using the user port of a BBC Micro model B computer [5]. Quantized maxima and minima were recorded for each sinusoid.

The 8 bit ADC uses successive approximation to digitize the signal and has a quoted error of ±1 LSB [5]. The effective input range of the ADC was measured as ±1.31V. The sampling rate achieved in combination with the computer was ≈225 samples per second. Therefore, by maintaining the generated sinusoid at a frequency of 10 Hz, the effects of spatial sampling, such as aliasing, were minimized.

In order to perform calculations, the input range of the ADC was represented as lying between 0 and 2.62V, thus $I_{MAX}$=2.62. This was also reflected in the calculation of the maxima and minima of the sinusoids, the mean level, *a*, of which were represented as 1.31V.

For each sinusoid, the input maxima and minima, $E(x)_{MAX}$ and $E(x)_{MIN}$, were calculated using the measured amplitude, *b*. The input signal modulation, $M_{IN}$ was also calculated. Using the quantization bitdepth, *d*, the quoted ADC error, *n*, and the measured variance of the input signal noise, $\sigma$, Equations 15, 16, and 17 were used to calculate $M_{QMAX}$, $M_{QMIN}$, and $\Delta M$. The actual quantized signal modulation was calculated from the measured quantized maxima and minima, $Q(x)_{MAX}$ and $Q(x)_{MIN}$, recorded by the computer.

### Results

Figure 7 shows measured quantized signal modulation versus input signal amplitude for the described system. Also shown are the envelope bounds, $M_{QMAX}$, $M_{QMIN}$, calculated using the above theory for the system in theoretically ideal and noisy conditions. Figure 7 shows the measured percentage change of the input signal modulation against input signal amplitude included with the calculated bounds.

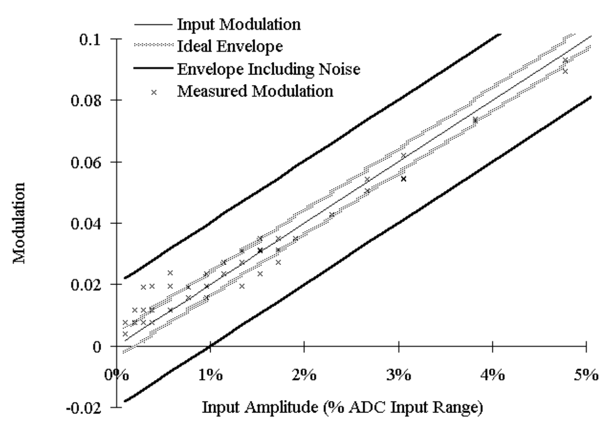


*Figure 7. Measured quantized signal modulation versus input signal amplitude for the Ferantti ZN449E 8 bit ADC. Also shown are the theoretical bounds calculated for an ideal and noisy system.*

The figures clearly show that a change in modulation exists due to the quantization of the signal. It is evident from Figure 8 that all measured points shown fall within the calculated theoretical bounds for the system when noise is included and a significant number within the bounds predicted for the ideal system. This suggests that the limits provided by the above theory are reasonable.

As the $2\sigma$ variance is used to describe the input signal noise, 5% of the noise values are excluded from the description. This leads to a slight chance that a quantized signal modulation value will fall outside of the calculated limits. This was not the case in this experiment.

Figure 8 shows that the percentage change in the input signal modulation increases as the input signal amplitude decreases. This agrees with the trend predicted by the theory. Significantly this will lead to an increase in error for high frequencies when using sinusoidal targets to measure MTF. This is due to the decreased modulation of those frequencies by components prior to the quantization stage.

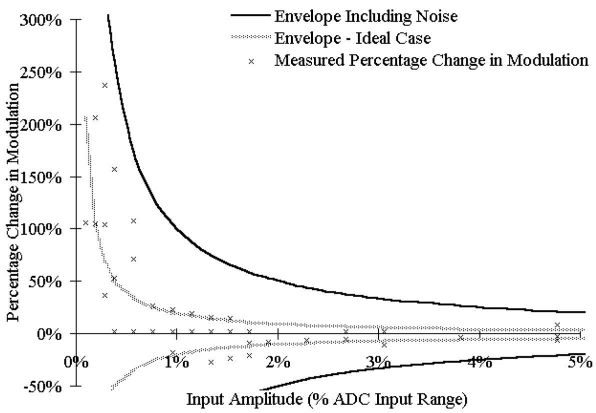


*Figure 8. Percentage change in the input signal modulation versus input signal amplitude for the 8 bit ADC. The theoretical bounds calculated for an ideal and noisy system are also shown.*

## Conclusions

A theoretical study has been performed upon the effect of quantization on modulation transfer function measurement in the absence of spatial sampling effects. The work clearly shows that there is a non-linear component in measured MTFs attributable to the quantization process when using sinusoidal test targets. This component is shown to be complex in nature, varying with respect to the input signal amplitude and quantization bitdepth. The general trend of this component is to increase as the quantization bitdepth and input signal amplitude decrease.

Formulae have been derived to calculate an envelope which predicts the bounds of the quantized signal modulation, given the input signal and quantization parameters, for ideal and noisy systems. Experimental results correlate well with the theoretical work suggesting validity of the description within the constraints given.

## Acknowledgments

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