Ellipse Fitting

The equation describing an ellipse can be written as

$$F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

with the constraint $b^2 - 4ac < 0$, where $a,\ b,\ c,\ d,\ e$, and f are the ellipse coefficients and (x, y) are the coordinates of the points on the ellipse. By defining $\mathbf{z}^\top = [x^2,\ xy,\ y^2,\ x,\ y,\ 1]$ and $\mathbf{h}^\top = [a,\ b,\ c,\ d,\ e,\ f]$, the equation can be written as

$$F(x, y) = \mathbf{h}^\top \mathbf{z} = 0. \tag{2}$$

The fitting of a conic to a set of points (x_i, y_i), $i = 1, ..., N$ may be seen as a problem of minimizing the function $\sum_{i=1}^{N} F(x_i, y_i)^2$ subject to the constraint $4ac - b^2 = 1$ that guarantees that the final solution is an ellipse. In matrix form, the problem can be rewritten as

$$\min_{\mathbf{h}} \|\mathbf{D}\mathbf{h}\|^2 \text{ subject to } \mathbf{h}^\top \mathbf{C}\mathbf{h} = 1, \tag{3}$$

where

$$\mathbf{D} = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ x_N^2 & x_N y_N & y_N^2 & x_N & y_N & 1 \end{bmatrix},$$

and

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This problem can be solved with the use of a Lagrange multiplier, which yields the following conditions for the parameter vector $\mathbf{h}$: $\mathbf{S}\mathbf{h} = \lambda \mathbf{C}\mathbf{h}$, with $\mathbf{h}^\top \mathbf{C}\mathbf{h} = 1$, where $\mathbf{S} = \mathbf{D}^\top \mathbf{D}$. It was also shown by Halif and Flusser[1] that the solution of the minimization problem is equivalent to finding a solution to the set of equations

$$\mathbf{M}\mathbf{h}_1 = \lambda \mathbf{h}_1, \tag{4}$$

$$\mathbf{h}_1^\top \mathbf{C}_1 \mathbf{h}_1 = 1, \tag{5}$$

$$\mathbf{h}_2 = -\mathbf{S}_3^{-1}\mathbf{S}_2^\top \mathbf{h}_1, \tag{6}$$

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}, \tag{7}$$

where $\mathbf{h}_1 = [a,\ b,\ c\,]^\top$, $\mathbf{h}_2 = [d,\ e,\ f\,]^\top$, $\mathbf{S}_1 = \mathbf{D}_1^\top \mathbf{D}_1$, $\mathbf{S}_2 = \mathbf{D}_1^\top \mathbf{D}_2$, $\mathbf{S}_3 = \mathbf{D}_2^\top \mathbf{D}_2$,

$$\mathbf{D}_1 = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ x_2^2 & x_2 y_2 & y_2^2 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_N y_N & y_N^2 \end{bmatrix}, \tag{8}$$

$$\mathbf{D}_2 = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{bmatrix}, \tag{9}$$

and $\mathbf{M}{=}\mathbf{C}_1^{-1}(\mathbf{S}_1{-}\mathbf{S}_2\mathbf{S}_3^{-1}\mathbf{S}_2^\top)$.

REFERENCES

1. R. Halif and J. Flusser, “Numerically stable direct least squares fitting of ellipses,” in *Proc. Sixth Int’l Conf. Computer Graphics and Visualization*, **1**, pp. 125 – 132, 1998.